DTMC

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Markov process

A stochastic process is called a Markov process when it has the Markov property:

$$P\{X_{t_n} \leq x_n | X_{t_{n-1}} = x_{n-1}, \dots, X_{t_1} = x_1\} = P\{X_{t_n} \leq x_n | X_{t_{n-1}} = x_{n-1}\}, \ \forall n, \forall t_1 < \cdots < t_n$$

- The future path of a Markov process, given its current state ($X_{t_{n-1}}$) and the past history before t_{n-1}, depends only on the current state (not on how this state has been reached).
- The current state contains all the information (summary of the past) that is needed to characterize the future (stochastic) behavior of the process.
- Given the state of the process at an instant its future and past are **conditionally** independent.

<u>Example</u> A process with independent increments is always a Markov process.

$$X_{t_n} = X_{t_{n-1}} + \underbrace{(X_{t_n} - X_{t_{n-1}})}$$

the increment is independent of all the previous increments
which have given rise to the state $X_{t_{n-1}}$

Markov chain

The use of the term Markov chain in the literature is ambiguous: it defines that the process is either a discrete time or a discrete state process.

In the sequel, we limit the use of the term for the case where the process is both discrete time and discrete state.

• Without loss of generality we can index the discrete instants of time by integers.

– A Markov chain is thus a process X_n, n = 0,1,....

• Similarly we can denote the states of the system by integers X_n = 0, 1, . . . (the set of states can be finite or countably finite).

In the following we additionally assume that the process is time homogeneous.
A Markov process of this kind is characterized by the (one-step) transition probabilities

(transition from state i to state j):

$$p_{i,j} = P\{X_n = j | X_{n-1} = i\}$$

time homogeneity: the transition probability does not depend on n

The probability of a path

The probability of a path i0,i1,…,in is

$$P\{X_0 = i_0, \ldots, X_n = i_n\} = P\{X_0 = i_0\} p_{i_0,i_1} p_{i_1,i_2} \cdots p_{i_{n-1},i_n}$$

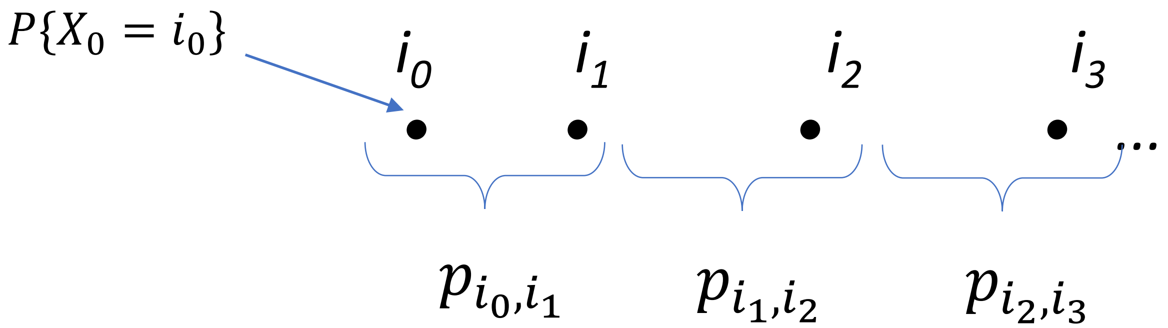
Proof

$$P\{X_0 = i_0, X_1 = i_1\} = \underbrace{P\{X_1 = i_1 | X_0 = i_0\}}_{p_{i_0,i_1}} P\{X_0 = i_0\}$$

$$P\{X_0 = i_0, X_1 = i_1, X_2 = i_2\} = \underbrace{P\{X_2 = i_2 | X_1 = i_1, X_0 = i_0\}}_{p_{i_1,i_2}} \underbrace{P\{X_1 = i_1, X_0 = i_0\}}_{p_{i_0,i_1} P\{X_0 = i_0\}}$$

$$= P\{X_0 = i_0\} p_{i_0,i_1} p_{i_1,i_2}$$

Similarly, the proof can be continued for longer sequences.



Dynamics of a MC: The transition probability matrix of a Markov chain

The transition probabilities can be arranged as transition probability matrix **P** = $(p_{i,j})$

Initial state →

$$\boldsymbol{P} = \begin{bmatrix} p_{0,0} & p_{0,1} & p_{0,2} & \cdots \\ p_{1,0} & p_{1,1} & p_{1,2} & \cdots \\ \vdots & \vdots & \vdots & \end{bmatrix}$$

↓ final state

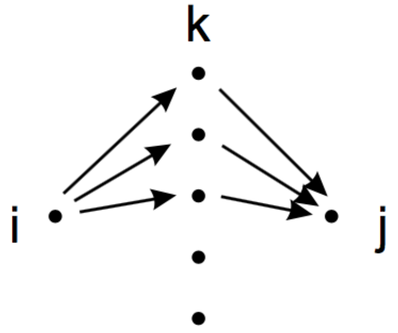
- The row i contains the transition probabilities from state i to other states.

– since the system always goes to some state, the sum of the row probabilities is 1

- A matrix with non-negative elements such that the sum of each row equals 1 is called a stochastic matrix.
- One can easily show that the product of two stochastic matrices is a stochastic matrix.

Many-step transition probability matrix

The probability that the system, initially in state i, will in state j after two steps is



$$\sum_k p_{i,k} p_{k,j}$$

(takes into account all paths via an intermediate state k).

Clearly this the element {i, j} of the matrix $\mathbf{P}^2$.

Similarly, one finds that the n-step transition probability matrix $\mathbf{P}^n$.

Denote its elements by $p_{i,j}^{(n)}$ (the subscript refers to the number of steps). Since it holds that

$\mathbf{P}^n = \mathbf{P}^m \cdot \mathbf{P}^{n-m}$ $(0 \leq m \leq n)$,we can write in component form

$p_{i,j}^{(n)} = \sum_k p_{i,k}^{(m)} p_{k,j}^{(n-m)}$ the Chapman-Kolmogorov equation

This simply expresses the law of total probability, where the transition in *n* steps from state *i* to state *j* is conditioned on the system being in state *k* after *m* steps.

State probabilities

Denote

$\pi_i^{(n)} = P\{X_n = i\}$ the probability that the process is in state i at time n

Arrange the state probabilities at time n in a state probability vector

$\boldsymbol{\pi}^{(n)}=(\pi_0^{(n)}, \pi_1^{(n)}, \pi_2^{(n)},\ldots)$

By the law of total probability we have

$P\{X_1 = i\}=\sum_k P\{X_1 = i|X_0 = k\}P\{X_0 = k\}$

or $\pi_i^{(1)} = \sum_k \pi_k^{(0)} p_{k,i}$

and in vector form $\boldsymbol{\pi}^{(1)} = \boldsymbol{\pi}^{(0)}\mathbf{P}$

As the process is Markovian and $\boldsymbol{\pi}^{(1)}$ represents the initial probabilities in the next step,

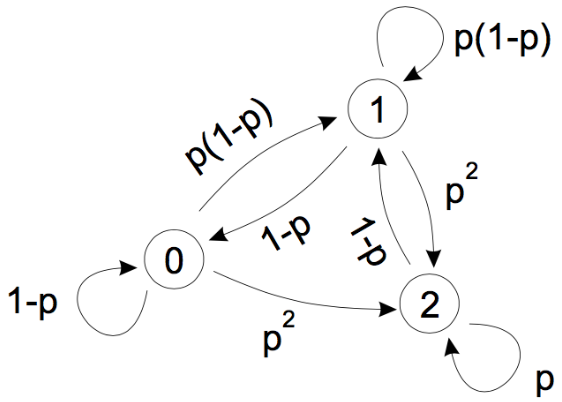
$\boldsymbol{\pi}^{(2)} = \boldsymbol{\pi}^{(1)}P$ and generally $\boldsymbol{\pi}^{(n)} = \boldsymbol{\pi}^{(n-1)}\mathbf{P}$

from which we have recursively

$\boldsymbol{\pi}^{(n)} = \boldsymbol{\pi}^{(0)}\mathbf{P}^n$ (Note, $\mathbf{P}^n$ is the n-step transition probability matrix.)

Example

$$\mathbf{P}=\begin{pmatrix} 1-p & p(1-p) & p^2 \\ 1-p & p(1-p) & p^2 \\ 0 & 1-p & p \end{pmatrix} \quad p = 1/3$$



$$\boldsymbol{P}^1=\frac{1}{9}\begin{pmatrix} 6 & 2 & 1 \\ 6 & 2 & 1 \\ 0 & 6 & 3 \end{pmatrix}= \begin{pmatrix} 0.6666 & 0.2222 & 0.1111 \\ 0.6666 & 0.2222 & 0.1111 \\ 0 & 0.6666 & 0.3333 \end{pmatrix}$$

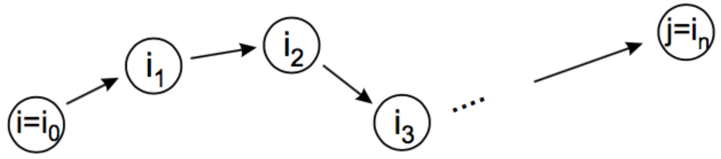
$$\boldsymbol{P}^2=\frac{1}{9^2}\begin{pmatrix} 48 & 22 & 11 \\ 48 & 22 & 11 \\ 36 & 30 & 15 \end{pmatrix}= \begin{pmatrix} 0.5926 & 0.2716 & 0.1358 \\ 0.5926 & 0.2716 & 0.1358 \\ 0.4444 & 0.3704 & 0.1852 \end{pmatrix}$$

$$\boldsymbol{P}^4=\frac{1}{9^3}\begin{pmatrix} 420 & 206 & 103 \\ 420 & 206 & 103 \\ 396 & 222 & 111 \end{pmatrix}= \begin{pmatrix} 0.5761 & 0.2826 & 0.1413 \\ 0.5761 & 0.2826 & 0.1413 \\ 0.5432 & 0.3045 & 0.1523 \end{pmatrix}$$

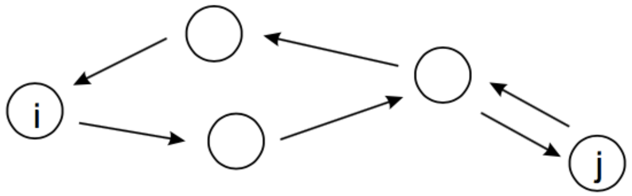
$$\boldsymbol{P}^8 = \begin{pmatrix} 0.5714 & 0.2857 & 0.1429 \\ 0.5714 & 0.2857 & 0.1429 \\ 0.5714 & 0.3057 & 0.1429 \end{pmatrix}$$

Starting from an initial state i, the distribution of the final state can be read from the row i. After 8 steps the final state distribution is independent of the initial state (to the accuracy of four digits): “the process forgets its initial state”.

Classification of states of a Markov chain



State *i* leads to state *j* (written $i \rightarrow j$), if there is a path $i_0 = i, i_1, \ldots, i_n = j$ such that all the transition probabilities are positive, $p_{i_k,i_{k+1}} > 0$, k = 0,…,n−1. Then $(\mathbf{P}^n)_{i,j} > 0$.



States i and j communicate (written $i \leftrightarrow j$), if $i \rightarrow j$ and $j \rightarrow i$.

Communication is an equivalence relation: the states can be grouped into equivalent classes

so that:

- within each class all the states communicate with each other
- two states from two different classes never communicate which each other

The equivalence classes defined by the relation $\leftrightarrow$ are called the irreducible classes of states

A Markov chain with a state space which is an irreducible class (the only one, i.e. all the states communicate) is called irreducible.

Classification of states (continued)

A set of states is closed, if none of its states leads to any of the states outside the set.

A single state which alone forms a closed set is called an absorbing state
- for an absorbing state we have $p_{i,i} = 1$
- one may reach an absorbing state from other states, but one cannot get out of it

Each state is either transient or recurrent.

- A state *i* is transient if the probability of returning to the state is < 1.
  i.e. there is a non-zero probability that the system never returns to the state.
- A state *i* is recurrent if the probability of returning to the state is = 1.
  i.e. with certainty, the system sometimes returns to the state.

Recurrent states are further classified according to the expectation of the time $T_{i,i}$ it takes to return to the state:

positive recurrent
expectation of first return time $< \infty$

null recurrent
expectation of first return time $= \infty$

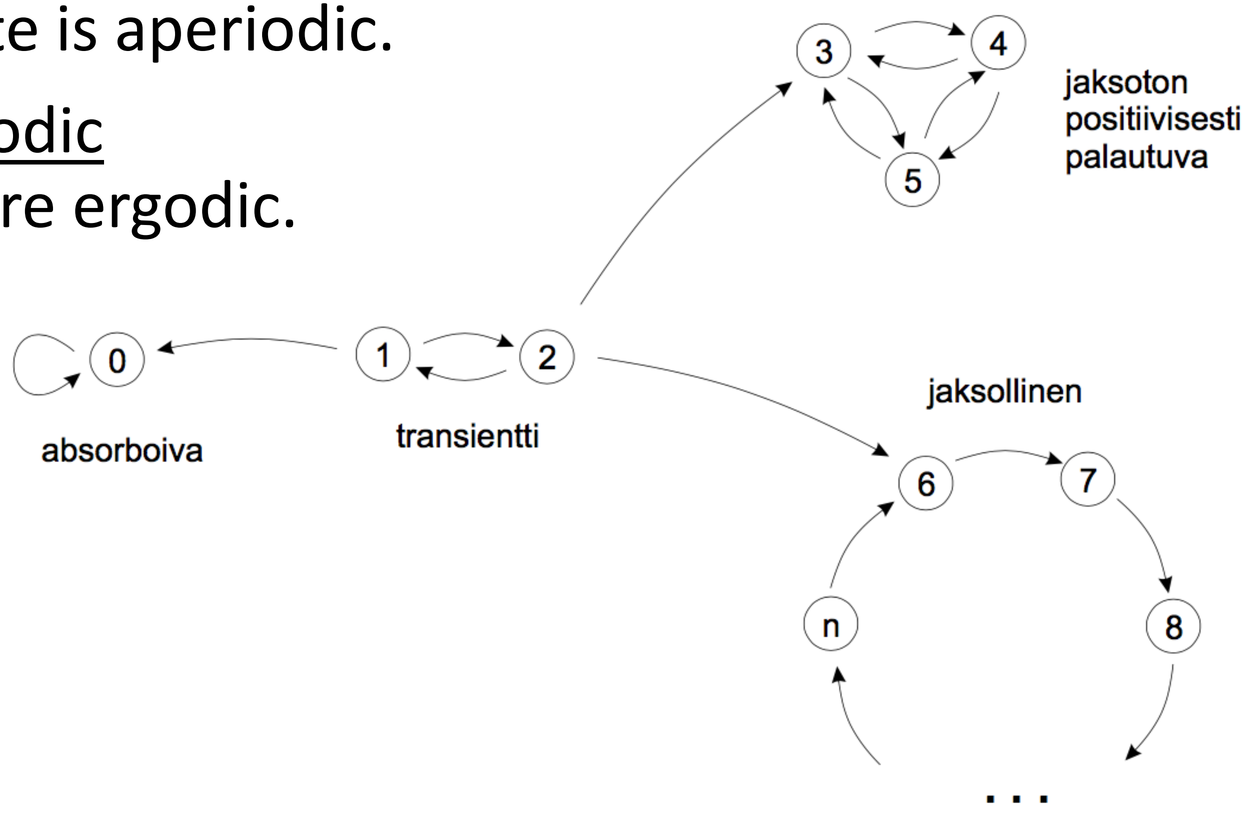
The first return time $T_{i,i}$ of state *i* is the time at which the Markov chain first returns to state i when $X_0 = i$.

Classification of states (continued)

Type	# of visits	$\mathrm{E}[T_{i,i}]$
Transient	$< \infty$	∞
Null recurrent	∞	∞
Positive recurrent	∞	$< \infty$

If the first return time of state i can only be a multiple of an integer d > 1 the state i is called <u>periodic</u>. Otherwise the state is aperiodic.

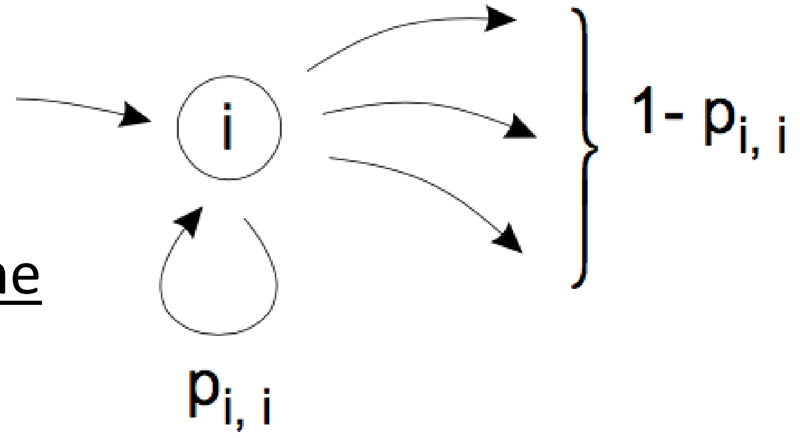
An aperiodic positive recurrent state is <u>ergodic</u>
A <u>Markov chain</u> is ergodic, iff all its states are ergodic.



Classification of states (continued)

Proposition: In an irreducible Markov chain either

- all the states are transient, or
- all the states are null recurrent, or
- all the states are positive recurrent



Remarks on the life time of a state and the first return time

The number of steps the system consecutively stays in state i is geometrically distributed

$\sim \text{Geom}(1 - p_{i,i})$

because the exit from the state occurs with the probability $1 - p_{i,i}$.

After each visit of state i, the first return time $T_{i,i}$ back to state i is independent of the first return times after any of the other visits to the state (follows from the Markov property).

Denote $\overline{T}_i = E[T_{i,i}]$

$$\overline{T}_i = \begin{cases} \infty & \text{if the state is transient or null recurrent} \\ < \infty & \text{if the state is positive recurrent} \end{cases}$$

Limiting distribution VS stationary distribution

Q: What happens to $p_{ij}^{(n)}$ as n goes to infinity?

Q: What is the lim $P^{(n)}$ as n->∞?

Q: Does it always converge? (we'll see this later)

- If limit exists, then $\lim_{n\to\infty} p_{ij}^{n} = \pi_j$ for any initial state i

- **π** = $\{\pi_0, \pi_1, \dots, \pi_m\}$ is called the stationary distribution

  IF π• P = π → $\pi_j = \sum \pi_i \cdot p_{ij}$

  and $\Sigma_i \pi_i = 1$

- The above equation can be used to find **π**

Kolmogorov's theorem

In an irreducible, aperiodic Markov chain there always exist the limits

$$\pi_j = \lim_{n\to\infty} \pi_j^{(n)} = \frac{1}{\bar{T}_j}$$

and these are independent of the initial state.

Furthermore, either

i) all the states of the chain are transient or all of the states are null recurrent; in either case $\pi_j = 0, \forall j$,

ii) all the states of the chain are positive recurrent, and there exists a unique stationary distribution $\boldsymbol{\pi}$ which is obtained as the solution of the equations

$\boldsymbol{\pi} = \boldsymbol{\pi} \cdot \mathbf{P}$ or $\pi_j = \sum_i \pi_i P_{i,j}$ and $\sum_j \pi_j = 1$

$\boldsymbol{\pi} \cdot \mathbf{e}^T = 1$

($\mathbf{e}$ is a row vector with all the components equal to 1, and $\mathbf{e}^T$ is the corresponding column vector)

Remarks on the stationary distribution

If the limit probabilities (the components of the vector) **π** exist, they must satisfy the equation **π** = **πP**, because

$$\boldsymbol{\pi} = \lim_{n\to\infty} \boldsymbol{\pi}^{(n)} = \lim_{n\to\infty} \boldsymbol{\pi}^{(n+1)} = \lim_{n\to\infty} \boldsymbol{\pi}^{(n)}.\mathbf{P} = \boldsymbol{\pi}.\mathbf{P}$$

The equation **π** = **πP** can also be expressed in the form: **π** is the (left) eigenvector of the matrix **P** belonging to the eigenvalue **1** (or belonging to the eigenvalue 0 of the matrix (**P**−**I**)).

π_j defines which proportion of time (steps) the system stays in state j.

In an irreducible, aperiodic Markov, the limit distribution **π** (aka steady state probabilities) is equal to the so called the <u>stationary distribution</u> or the <u>equilibrium distribution</u>

Note. An equilibrium does not mean that nothing happens in the system, but merely that the information on the initial state of the system has been “forgot” or “washed out” because of the stochastic development.

Global balance

Equation **π** = **πP** or $\pi_j = \sum_i \pi_i P_{i,j}$ ∀j, is often called the (global) balance condition.

Since for row *j*; $\sum_i P_{j,i}$ =1 (the transition takes the system to some state), one can write

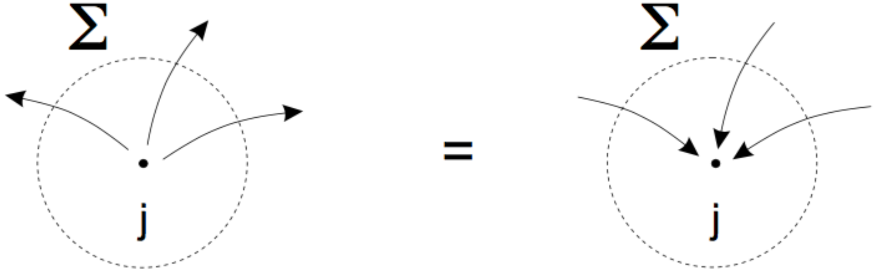
$$\sum_i \pi_j P_{j,i} \quad = \quad \sum_i \pi_i P_{i,j}$$

One equation for each state j.

prob. that the system is in state j and makes a transition to another state

prob. that the system is in another state and makes a transition to state j

Balance of probability flows: there are as many exits form state j as there are entries to it.



If the balance equations are known to be satisfied for all but one of the states, they are automatically satisfied also for that particular state (due to conservation of probability flows)

- – the balance equations are linearly dependent
  (⇒ the homogeneous equation **π** = **πP** has a non-zero solution)
- – the solution is determined up to a constant factor
- – in order to determine the unknown factor, one needs the normalization condition $\sum_j \pi_j$ =1

Example. We revisit the previous example (now with a general p).

$$(\pi_0 \quad \pi_1 \; \pi_2) = (\pi_0 \quad \pi_1 \quad \pi_2)\begin{pmatrix} 1-p & p(1-p) & p^2 \\ 1-p & p(1-p) & p^2 \\ 0 & 1-p & p \end{pmatrix}$$

Write the first two equations (equalities of the first two components of the vectors on the lhs and rhs)

$$\pi_0 = (1-p)\pi_0 + (1-p)\pi_1 \Rightarrow \pi_0 = \frac{1-p}{p}\pi_1 \qquad p\pi_0 = (1-p)\pi_1$$

$$\pi_1 = p(1-p)\pi_0 + p(1-p)\pi_1 + (1-p)\pi_2 = (1-p)^2\pi_1 + p(1-p)\pi_1 + (1-p)\pi_2$$

$$= (1-p)\pi_1 + (1-p)\pi_2$$

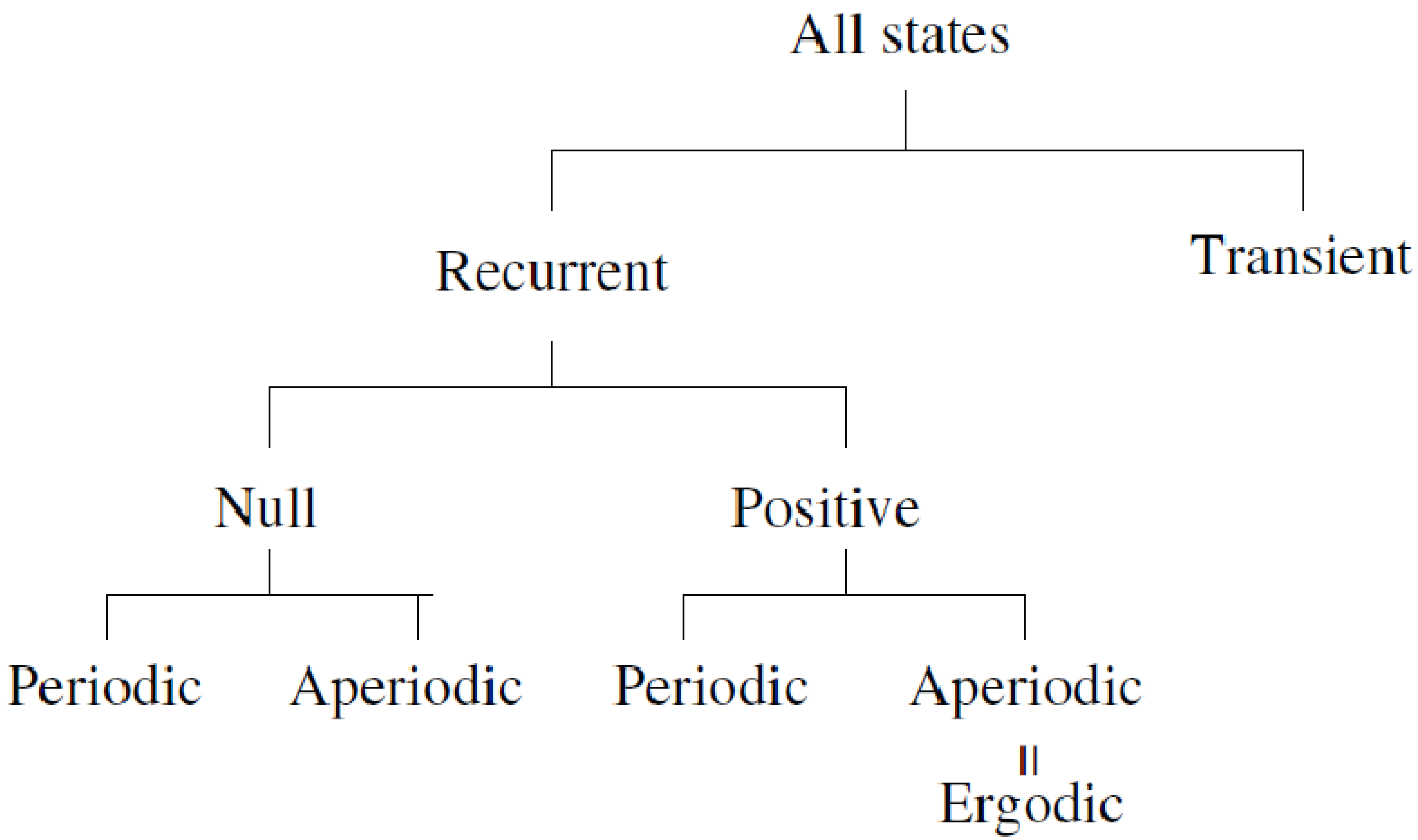
$$\Rightarrow \pi_1 = \frac{1-p}{p}\pi_2 \quad \Rightarrow \quad \pi_0 = \left(\frac{1-p}{p}\right)^2 \pi_2$$

$$\boldsymbol{\pi} = \left(\left(\frac{1-p}{p}\right)^2 \quad \frac{1-p}{p} \quad 1\right)\pi_2$$

By the normalization condition $\pi_0 + \pi_1 + \pi_2 = 1$ one gets

$$\boldsymbol{\pi} = \left(\frac{(1-p)^2}{1-p(p-1)} \quad \frac{p(1-p)}{1-p(p-1)} \quad \frac{p^2}{1-p(p-1)}\right)$$ With $p = \frac{1}{3}$: $\boldsymbol{\pi} = (\,0.5714 \quad 0.2857 \quad 0.1429\,)$

15.3 Classification of States



Calculation of Stationary Distribution

A. Finite number of states

- Solve explicitly the system of equations

$$\pi_j = \sum_{i=0}^{m} \pi_i P_{ij}, \quad j = 0, 1, \ldots, m$$

$$\sum_{i=0}^{m} \pi_i = 1$$

- Numerically from P^n which converges to a matrix with rows equal to π
- ➧ Suitable for a small number of states

B. Infinite number of states

- Cannot apply previous methods to problem of infinite dimension
- Guess a solution to recurrence:

$$\pi_j = \sum_{i=0}^{\infty} \pi_i P_{ij}, \quad j = 0, 1, \ldots,$$

$$\sum_{i=0}^{\infty} \pi_i = 1$$

See appendix

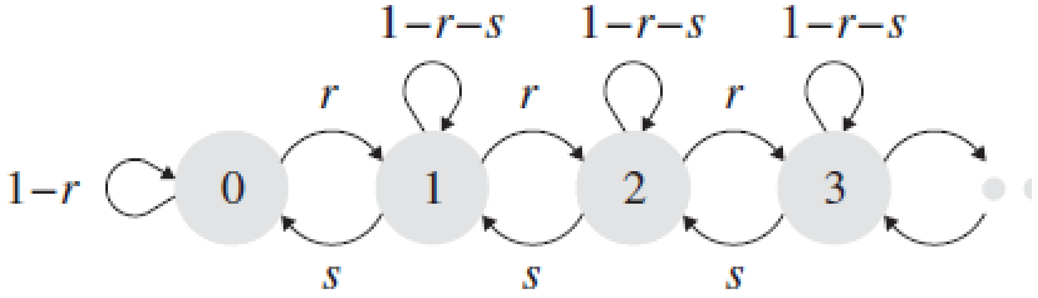
Solving Stationary Equations in Infinite-State DTMCs

- Consider an **unbounded queue** that at every time step, with probability p = 1/40 one job arrives, and independently, with probability q = 1/30 one job departs.
- *what is the average number of jobs in the system?*
- we model the problem as a DTMC with an infinite number of states: 0, 1, 2, . . . , representing the number of jobs at the router.
- Let $r = p(1 - q)$ =29/1200 and $s = q(1 - p)$ = 39/1200 , where $r < s$.

r= one enters, no exit, s= one exits, no enter

Solving Stationary Equations in Infinite-State DTMCs

- the DTMC for the unbounded Queue and the transition probability matirx



$$
\mathbf{P} = \begin{pmatrix} 1-r & r & 0 & 0 & \dots \\ s & 1-r-s & r & 0 & \dots \\ 0 & s & 1-r-s & r & \dots \\ 0 & 0 & s & 1-r-s & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}
$$

Solving Stationary Equations in Infinite-State DTMCs

- the stationary equations:

$$
\begin{aligned}
\pi_0 &= \pi_0(1-r) + \pi_1 s \\
\pi_1 &= \pi_0 r + \pi_1(1-r-s) + \pi_2 s \\
\pi_2 &= \pi_1 r + \pi_2(1-r-s) + \pi_3 s \\
\pi_3 &= \pi_2 r + \pi_3(1-r-s) + \pi_4 s \\
&\vdots
\end{aligned}
$$

$$\pi_0 + \pi_1 + \pi_2 + \pi_3 + \cdots = 1$$

Solving Stationary Equations in Infinite-State DTMCs

- **Q:** How we solve this infinite number of equations?
- the first equation can be rewritten as

$$\pi_1 = \frac{r}{s}\pi_0$$

- And π_2 in terms of π_0

$$\pi_2 = \left(\frac{r}{s}\right)^2 \pi_0$$

- And make a general guess

$$\pi_i = \left(\frac{r}{s}\right)^i \pi_0$$

Solving Stationary Equations in Infinite-State DTMCs

- To verify your guess, you need to showthat it satisfies the stationary equations

$$\pi_i = \pi_{i-1} r + \pi_i (1 - r - s) + \pi_{i+1} s$$

$$\left(\frac{r}{s}\right)^i \pi_0 = \left(\frac{r}{s}\right)^{i-1} \pi_0 r + \left(\frac{r}{s}\right)^i \pi_0 (1 - r - s) + \left(\frac{r}{s}\right)^{i+1} \pi_0 s \quad \checkmark$$

- Using normalization equation $\sum_i \pi_i$

$$\pi_0 \cdot \left(1 + \frac{r}{s} + \left(\frac{r}{s}\right)^2 + \left(\frac{r}{s}\right)^3 + \cdots\right) = 1$$

$$\pi_0 \cdot \left(\frac{1}{1 - \frac{r}{s}}\right) = 1$$

$$\pi_0 = 1 - \frac{r}{s}.$$

$$\pi_i = \left(\frac{r}{s}\right)^i \cdot \left(1 - \frac{r}{s}\right).$$

Solving Stationary Equations in Infinite-State DTMCs

- **Q:** What is the average number of jobs at the server?
- **A:** Let *N* denote the number of jobs at the server. Then
- $E[n]=0.\ \pi_0+1.\ \pi_1+2.\ \pi_2+\ldots$
- Define

$$\rho = \frac{r}{s}$$

- Then $\pi_i=\rho^i(1-\rho)$

Solving Stationary Equations in Infinite-State DTMCs

- Thus

$$
\begin{aligned}
\mathbf{E}[N] &= 1\rho(1-\rho) + 2\rho^2(1-\rho) + 3\rho^3(1-\rho) + \ldots \\
&= (1-\rho)\rho\left(1 + 2\rho + 3\rho^2 + 4\rho^3 + \ldots\right) \\
&= (1-\rho)\rho\frac{d}{d\rho}\left(1 + \rho + \rho^2 + \rho^3 + \rho^4 + \ldots\right) \\
&= (1-\rho)\rho\frac{d}{d\rho}\left(\frac{1}{1-\rho}\right) \\
&= (1-\rho)\rho \cdot \frac{1}{(1-\rho)^2} \\
&= \frac{\rho}{1-\rho}
\end{aligned}
$$

- For our example $\rho = 29/39$ and $\mathbf{E}[N] = 2.9$. So on average there are about 3 jobs in the system.